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APPROXIMATION OF ARBITRATION GAMES BY FINITE SUBGAMES

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SUMMARY. In this paper two-person cooperative games are considered, where an arbitrator, using a bargaining solution which is known to both players, assigns a Pareto optimal point in the cooperative payoff region to each non-cooperative payoff pair. Although there do not exist continuous bargaining solutions, it appears possible to derive theorems about the existence of arbitration values with the aid of a theorem of H. Raiffa, by approximating such arbitration games by finite subgames.

1. Introduction

Arbitration games were introduced independently by Raiffa (1953) and Nash (1953).

In the definition of an arbitration game the notions two-person game in normal form and bargaining solution play a role.

A *two-person game in normal form* is a quadruplet $< X, Y, K_1, K_2 >$, where X is a non-empty set (the strategy space of player 1), Y is a non-empty set (the strategy space of player 2) and $K_i : X \times Y \rightarrow \mathfrak{R}$ is, for $i \in \{1, 2\}$, a bounded realvalued function on the Cartesian product of the strategy spaces (the payoff function for player i).

A bargaining solution is a map ϕ from the set of bargaining pairs

$$\mathcal{B} := \{(a, S);\ S \text{ is a compact convex subset of } \mathfrak{R}^2,\ a \in S\}$$

into $\mathfrak{R}^2$, with the property that

$$\phi(a, S) \geqslant a,\ \phi(a, S) \in \mathcal{P}(S), \text{ for each } (a, S) \in \mathcal{B},$$

where

$$\mathcal{P}(S) := \{p \in S;\ \text{for each } s \in S \text{ with } s \geqslant p, \text{ we have } s = p\},$$

is the *Pareto set* of S.

Now let $\Gamma = < X, Y, K_1, K_2 >$ be a two-person game in normal form. The set

$$\mathcal{R}_0(\Gamma) := \{K(x, y) = (K_1(x, y), K_2(x, y)) \in \mathfrak{R}^2 ;\ (x, y) \in X \times Y\}$$

is the set of possible payoff pairs, if there is no cooperation. If the players may use correlated strategies i.e. (discrete) probability measures on $X \times Y$, then each point of the *cooperative payoff region*

$$\mathcal{R}(\Gamma) := \text{cl}\,(\text{conv}\,(\mathcal{R}_0)(\Gamma))$$

can be approached as near as the players want. Let us denote the Pareto set of $\mathcal{R}(\Gamma)$ by $\mathcal{P}(\Gamma)$. Of course, the players want to reach a payoff point in $\mathcal{P}(\Gamma)$, but the problem is which one.

A bargaining solution $\phi : \mathcal{B} \to \mathfrak{R}^2$ determines an *arbitration function* $\phi^\Gamma : \mathcal{R}(\Gamma) \to \mathcal{P}(\Gamma)$, where

$$\phi^\Gamma(r) := \phi(r, \mathcal{R}(\Gamma)), \text{ for all } r \in \mathcal{R}(\Gamma).$$

In the following we will often write $\phi(r)$ instead of $\phi^\Gamma(r)$.

Given Γ and ϕ, we look at the situation, where the players who have to play Γ, decide to solve their problem with the aid of an arbitrator, who uses the bargaining solution ϕ as follows.

Step 1 : Independently of each other, the players assign an $x^0 \in X$ and a $y^0 \in Y$, respectively, and deliver it to the arbitrator.

Step 2 : The arbitrator calculates the payoff $\phi(K_1(x^0, y^0), K_2(x^0, y^0))$ and chooses a correlated strategy μ, such that the expected payoff $\int\int K(x, y)d\mu(x, y)$ with respect to μ equals $\phi(K(x^0, y^0))$, if that is possible, otherwise μ is chosen, such that $\int\int K(x, y)d\mu(x, y)$ is as close to $\phi(K(x^0, y^0))$ as both players want.

Step 3 : With a lottery corresponding to μ, an outcome $(x^1, y^1) \in X \times Y$ is determined.

Step 4 : The non-cooperative game Γ is played, where players 1 and 2 are obliged to choose x^1 and y^1, respectively, resulting in a payoff $(K_i(x^1, y^1)$, for player $i \in \{1, 2\}$.

From a strategic point of view, for the players, this new game is, essentially, the non-cooperative game in normal form

$$\Gamma_\phi = < X, Y, \phi_1 K, \phi_2 K >,$$

where $\phi_i K(x, y)$ is the i-th coordinate of $\phi(K_1(x, y), K_2(x, y)) \in \mathfrak{R}^2$. Γ_ϕ is called the *arbitration game*, corresponding to the game Γ and the bargaining solution ϕ.

Definition 1.1 : Let Γ_ϕ be an arbitration game. We say that Γ_ϕ is *strictly determined* if

$$(v_1(\Gamma_\phi), v_2(\Gamma_\phi)) := (\sup_{x \in X} \inf_{y \in Y} \phi_1 K(x, y), \sup_{y \in Y} \inf_{x \in X} \phi_2 K(x, y)) \in \mathcal{P}(\Gamma).$$

In that case, $v_i(\Gamma_\phi)$ is called the *arbitration value* for player i and $v(\Gamma_\phi) = (v_1(\Gamma_\phi), v_2(\Gamma_\phi))$ the *value* of the arbitration game. If Γ_ϕ is strictly determined, the elements of

$$O_1(\Gamma_\phi) := \{\hat{x} \in X;\ \inf_{y \in Y} \phi_1\, K(\hat{x}, y) = v_1(\Gamma_\phi)\}$$

are called *optimal strategies for player* 1. The set

$$O_2(\Gamma_\phi) := \{\hat{y} \in Y;\ \inf_{x \in X} \phi_2\, K(x, \hat{y}) = v_2(\Gamma_\phi)\}$$

is the *optimal strategy space of player* 2.

The purpose of this paper is to prove the existence of arbitration values for some classes of games by approximating those games by sequences of subgames for which it is known that the arbitration value exists.

In other fields of game theory this approximation procedure was also successful [cf. Wald (1945), Tijs (1975) and Owen (1976)]. However, the problems to overcome in the field of arbitration games are greater because there do not exist continuous bargaining solutions (cf. Jansen and Tijs, 1980, proposition 3.1). The relation between the value of the game and the values of the approximating subgames is also more complicated.

Recall that $\mathcal{B}$ is provided with the topology, induced by the metric $d : \mathcal{B} \times \mathcal{B} \to \Re$, defined by

$$d((a, S), b, T)) := \max \{\|a-b\|_\infty, d_H(S, T)\}, \text{ for all } (a, S), (b, T) \in \mathcal{B},$$

with

$$\|a-b\|_\infty := \max \{|a_1-b_1|, |a_2-b_2|\}$$

and

$$d_H(S, T) := \inf \{\varepsilon > 0 ;\ S \subset B_\varepsilon(T), T \subset B_\varepsilon(S)\}$$

is the Hausdorff distance between S and T

$$(\text{where } B_\varepsilon(S) := \{x \in \Re^2 ;\ \inf_{s \in S} \|x-s\|_\infty \leqslant \varepsilon\}).$$

Since most bargaining solutions proposed in the literature are upper semicontinuous (cf. Jansen and Tijs, 1980, propositions 3.2, 3.3, 3.4 and 3.5), we will restrict our attention in this paper to upper semicontinuous bargaining solutions.

Definition 1.2 : Let $\phi : \mathcal{B} \rightarrow \Re^2$ be a bargaining solution. We shall say that ϕ is *upper semicontinuous* (u.s.c.) if for each sequence

$$(a,\ S),\ (a(1),\ S_1),\ (a(2),\ S_2),\ \dots \text{ in } \mathcal{B} \text{ with } \lim_{n\rightarrow\infty}\ (a(n),\ S_n) = (a,\ S),$$

we have

$$\phi_i(a\ S) \geqslant \limsup_{n\rightarrow\infty}\ \phi_i(a(n)S_n), \text{ for } i \in \{1,\ 2\}.$$

We shall call ϕ *regular* if for each $(a\ S), (b,\ S) \in \mathcal{B}$ with $\phi(a,\ S) = \phi(b,\ S)$, we have $\phi(\lambda a+(1-\lambda)b,\ S) = \phi(a,\ S)$, for all $\lambda \in [0,\ 1]$.

The bargaining solutions proposed by J. F. Nash (1950) and Kalai and Rosenthal (1978) are u.s.c. and regular. We note that for each u.s.c. bargaining solution and each two-person game Γ the corresponding arbitration function ϕ^Γ is continuous (cf. Tijs and Jansen, 1980, p. 7, see also Lemma 4.2 in this paper).

For later use we recall here the existence theorem of H. Raiffa (1953) in a modified form, which was proved with the aid of Kakutani's fixed point theorem for multifunctions.

Theorem 1.3 (Raiffa) : *Let* $\phi : \mathcal{B} \rightarrow \Re^2$ *be an u.s.c. and regular bargaining solution. Let* $\widetilde{\Gamma}$ *be the mixed extension of the game* $\Gamma = < X, Y, K_1, K_2 >$ *with finite strategy spaces* X *and* Y. *Then* $\widetilde{\Gamma}_\phi$ *possesses an arbitration value and the optimal strategy spaces* $O_1(\widetilde{\Gamma}_\phi)$ *and* $O_2(\widetilde{\Gamma}_\phi)$ *are non-empty sets.*

With the aid of this theorem, we derive, in Section 3, an existence theorem for continuous arbitration games and, in Section 4, for semi-infinite arbtiration games. Some other tools are introduced in Section 2.

2. Sequences of subgames

We start with some notation. Let $\Gamma = < X, Y, K_1, K_2 >$ be a two-person game in normal form. Then

$$\mathcal{W}(\Gamma) := \{r \in \mathcal{R}(\Gamma);\ \text{for each } s \in \mathcal{R}(\Gamma) \text{ with } s \geqslant r, \text{ we have } s_1 = r_1 \text{ or } s_2 = r_2\}$$

($\mathcal{W}(\Gamma)$) is called the *weak Pareto boundary* of $\mathcal{R}(\Gamma)$)

$$\bar{p}(\Gamma) = \left(\min_{p \in \mathcal{P}(\Gamma)} p_1,\ \max_{p \in \mathcal{P}(\Gamma)} p_2\right),\ \underline{p}(\Gamma) = \left(\max_{p \in \mathcal{P}(\Gamma)} p_1,\ \min_{p \in \mathcal{P}(\Gamma)} p_2\right)$$

$$\overline{\mathcal{W}}(\Gamma) = \{w \in \mathcal{W}(\Gamma);\ w_2 = \bar{p}_2(\Gamma)\},\ \underline{\mathcal{W}}(\Gamma) = \{w \in \mathcal{W}(\Gamma);\ w_1 = \underline{p}_1(\Gamma)\}$$

$j_\Gamma : \mathcal{W}(\Gamma) \to \mathcal{P}(\Gamma)$ is the map, defined by

$$j_\Gamma(w) = \begin{cases} \bar{\bar{p}}(\Gamma) & w \in \overline{\mathcal{W}}(\Gamma) \\ w \quad \text{if} & w \in \mathcal{P}(\Gamma) \\ \underline{p}(\Gamma) & w \in \underline{\mathcal{W}}(\Gamma). \end{cases}$$

In proposition 2.2, a sequence of games plays a role, in which each element $\Gamma^n = \langle X^n, Y^n, K_1^n, K_2^n \rangle$ of the sequence is a *subgame* of a fixed game $\Gamma = \langle X, Y, K_1, K_2 \rangle$ i.e. $X^n \subset X$, $Y^n \subset Y$ and $K_i^n : X^n \times Y^n \to \mathfrak{R}$ is the restriction of K_i to $X^n \times Y^n$, for $i \in \{1, 2\}$.

In the proof of proposition 2.2 we need the following

Lemma 2.1 : *Let (a, S), $(a(1), S_1)$, $(a(2), S_2)$, ... be a sequence in $\mathcal{B}$ converging to (a, S). Suppose that $a(n) \in \mathcal{P}(S_n)$, for all $n \in \mathcal{N}$. Then a is an element of*

$\mathcal{W}(S) := \{w \in S;$ *for each* $s \in S$ *with* $s \geqslant w$, *we have* $s_1 = w_1$ *or* $s_2 = w_2\}$.

Proof : The lemma is a direct consequence of Lemma 2.1 in Jansen and Tijs (1980). □

Proposition 2.2 : *Let $\Gamma^1, \Gamma^2, \Gamma^3, \ldots$ be a sequence of subgames of Γ and let ϕ be an u.s.c. bargaining solution.*

Suppose, furthermore, that the following properties hold :

(i) $\lim\limits_{n \to \infty} d_H(\mathcal{R}(\Gamma^n), \mathcal{R}(\Gamma)) = 0$,

(ii) *for each $n \in \mathcal{N}$, the arbitration game Γ^n_ϕ possesses an arbitration value,*

(iii) $v_i(\Gamma_\phi) \geqslant \limsup\limits_{n \to \infty} v_i(\Gamma^n_\phi)$, *for* $i \in \{1, 2\}$.

Then Γ_ϕ is strictly determined and $v(\Gamma_\phi) = j_\Gamma(\lim\limits_{n \to \infty} v(\Gamma^n_\phi))$.

Proof : In Tijs and Jansen (1979, Lemma 4.1), it was proved that $v(\Gamma_\phi) \in \operatorname{conv}(\{z\} \cup \mathcal{P}(\Gamma))$, where $z = \left(\min\limits_{p \in \mathcal{P}(\Gamma)} p_1, \min\limits_{p \in \mathcal{P}(\Gamma)} p_2\right)$ and where conv denotes convex hull. By properties (i) and (ii) and Lemma 2.1, $\limsup\limits_{n \to \infty} v(\Gamma^n_\phi) \in \mathcal{W}(\Gamma)$. Then, by property (iii), $v(\Gamma_\phi) \in \mathcal{W}(\Gamma)$. Since $\mathcal{W}(\Gamma) \cap \operatorname{conv}(\{z\} \cup \mathcal{P}(\Gamma)) = \mathcal{P}(\Gamma)$, we have $v(\Gamma_\phi) \in \mathcal{P}(\Gamma)$; hence Γ_ϕ is strictly determined. It is straightforward to prove that the facts $v(\Gamma_\phi) \in \mathcal{P}(\Gamma)$, $\limsup\limits_{n \to \infty} v(\Gamma^n_\phi) \in \mathcal{W}(\Gamma)$ and (iii) imply that $v(\Gamma_\phi) = j_\Gamma(\lim\limits_{n \to \infty} v(\Gamma^n_\phi))$. □

This proposition is very useful in the derivation of existence theorems. We will illustrate this in the following two sections for two classes of arbitration games.

3. Approximation of continuous arbitration games by finite subgames

In this section, $\phi : \mathcal{B} \rightarrow \Re^2$ is an u.s.c. regular bargaining solution and $\Gamma = < X, Y, K_1, K_2 >$ a *continuous two-person game* in normal form i.e., the strategy spaces X and Y are compact metric spaces and the payoff functions K_1 and K_2 are continuous realvalued functions on $X \times Y$. The *mixed extension* $\widetilde{\Gamma}$ of Γ is the two-person game in normal form $< \widetilde{X}, \widetilde{Y}, \widetilde{K}_1, \widetilde{K}_2 >$, where $\widetilde{X}$ ($\widetilde{Y}$) is the family of all probability measures on X (Y) and

$$\widetilde{K}_i(\mu, \nu) = \iint K_i(x, y) d\mu(x) d\nu(y), \text{ for all } \mu \in \widetilde{X}, \nu \in \widetilde{Y} \text{ and } i \in \{1, 2\}.$$

We want to prove that the arbitration game $\widetilde{\Gamma}_\phi$ possesses an arbitration value and optimal strategies for both players, using a sequence $\widetilde{\Gamma}^1, \widetilde{\Gamma}^2, \ldots$ of mixed extensions of finite subgames of Γ. The subgames are constructed as follows. For each $n \in \mathcal{N}$, there exists a partition of X into a finite number of non-empty Borel measurable subsets $C_1(n), C_2(n), \ldots, C_{p(n)}(n)$, where the diameter of each of these subsets is at most $\frac{1}{n}$. Similarly, there exists a Borel measurable partition $D_1(n), D_2(n), \ldots, D_{q(n)}(n)$ of Y with diameter $\text{diam}(D_j(n)) \leqslant \frac{1}{n}$ for each $j \in \{1, 2, \ldots, q(n)\}$. For $i \in \{1, 2, \ldots, p(n)\}$, take $x_i(n) \in C_i(n)$ and for $j \in \{1, 2, \ldots, q(n)\}$ take $y_j(n) \in D_j(n)$. Then $\widetilde{\Gamma}^n$ is the mixed extension of the finite game $\Gamma^n = < X^n, Y^n, K_1^n, K_2^n >$, where

$$X^n := \{x_1(n), x_2(n), \ldots, x_{p(n)}(n)\},$$

$$Y^n := \{y_1(n), y_2(n), \ldots, y_{q(n)}(n)\}$$

and K_1^n and K_2^n are the restrictions of K_1 and K_2 to $X^n \times Y^n$. We denote in the following the arbitration value of $\widetilde{\Gamma}_\phi^n$ by v^n. This value exists in view of Raiffa's Theorem 1.3.

Theorem 3.1 : *Let $\widetilde{\Gamma}, \widetilde{\Gamma}^n, \phi, v^n$ be as above. Then*

(i) *$\widetilde{\Gamma}_\phi$ possesses an arbitration value v,*

(ii) $v = j_{\widetilde{\Gamma}}(\limsup_{n \to \infty} v^n)$,

(iii) *$O_i(\widetilde{\Gamma}_\phi) \neq \phi$, for $i \in \{1, 2\}$.*

Proof: (a) First we prove that $\lim_{n \to \infty} d_H(\mathcal{R}(\widetilde{\Gamma}^n), \mathcal{R}(\Gamma)) = 0$. Note that

$$\mathcal{R}_0(\widetilde{\Gamma}^n) \subset \mathcal{R}_0(\widetilde{\Gamma}), \mathcal{R}(\widetilde{\Gamma}^n) \subset \mathcal{R}(\widetilde{\Gamma}), \text{ for each } n \in \mathcal{N}, \qquad \ldots \quad (3.1)$$

because $\widetilde{X}^n \subset \widetilde{X}$ and $\widetilde{Y}^n \subset \widetilde{Y}$. Take $\varepsilon > 0$ and $\mu \in \widetilde{X}$, $\nu \in \widetilde{Y}$. For each $n \in \mathcal{N}$, let $\sigma_n(\mu)$ be the probability measure $\sum_{i=1}^{p(n)} \mu(C_i(n))\delta(x_i(n))$ and $\tau_n(v) := \sum_{j=1}^{q(n)} \nu(D_j(n))\delta(y_j(n))$. Here $\delta(x)$ is the probability measure with mass 1 in x.

Since K_1 and K_2 are uniform continuous functions on $X \times Y$, there exists an $n^* \in \mathcal{N}$ such that, for all $n \geqslant n^*$ and $i \in \{1, 2\}$,

$$|\widetilde{K}_i(\mu, \nu) - \widetilde{K}_i^n(\sigma_n(\mu), \tau_n(\nu))| \leqslant \varepsilon. \qquad \ldots \quad (3.2)$$

In view of (3.1) and (3.2), we have $d_H(\mathcal{R}_0(\widetilde{\Gamma}^n), \mathcal{R}_0(\widetilde{\Gamma})) \leqslant \varepsilon$ and then also $d_H(\mathcal{R}(\widetilde{\Gamma}^n), \mathcal{R}(\widetilde{\Gamma})) \leqslant \varepsilon$, for all $n \geqslant n^*$. Hence $\lim_{n \to \infty} d_H(\mathcal{R}(\widetilde{\Gamma}^n), \mathcal{R}(\widetilde{\Gamma})) = 0$.

(b) Now we show that

$$v_i(\widetilde{\Gamma}_\phi) \geqslant \limsup_{n \to \infty} v_i^n, \text{ for } i \in \{1, 2\}. \qquad \ldots \quad (3.3)$$

We give the proof for $i = 1$. In view of Raiffa's Theorem 1.3, we can take $\hat{\mu}^n \in O_1(\widetilde{\Gamma}^n_\phi)$ and $\hat{\nu}^n \in O_2(\widetilde{\Gamma}^n_\phi)$. Since $\widetilde{X}$ is a compact metric space (cf. Billingsley, 1968, p. 240), there exists a subsequence $n(1), n(2), \ldots$, of $1, 2, \ldots$ such that the sequence of probability measures $\hat{\mu}^{n(1)}, \hat{\mu}^{n(2)}, \ldots$ on X and the sequence $\hat{\nu}^{n(1)}, \hat{\nu}^{n(2)}, \ldots$ of probability measures on Y weakly converge, say, to $\hat{\mu}$ and $\hat{\nu}$, respectively and such that

$$\lim_{k \to \infty} v^{n(k)} = \limsup_{n \to \infty} v^n. \qquad \ldots \quad (3.4)$$

Now take an arbitrary $\nu \in \widetilde{Y}$. Then

$$\lim_{k \to \infty} (\widetilde{K}_1^{n(k)}(\hat{\mu}^{n(k)}, \tau_{n(k)}(\nu)), \widetilde{K}_2^{n(k)}(\hat{\mu}^{n(k)}, \tau_{n(k)}(\nu)) = (\widetilde{K}_1(\hat{\mu}, \nu), \widetilde{K}_2(\hat{\mu}, \nu)). \quad \ldots \quad (3.5)$$

By (a) of the proof, (3.5) and the upper semicontinuity of ϕ_1 we obtain

$$\phi_1 \widetilde{K}(\hat{\mu}, \nu) \geqslant \limsup_{k \to \infty} \phi_1 \widetilde{K}^{n(k)}(\hat{\mu}^{n(k)}, \tau_{n(k)}(\nu)) \geqslant \lim_{k \to \infty} v_1^{n(k)}. \qquad \ldots \quad (3.6)$$

But then

$$v_1(\widetilde{\Gamma}_\phi) \geqslant \inf_\nu \phi_1 \widetilde{K}(\hat{\mu}, \nu) \geqslant \lim_{k \to \infty} v_1^{n(k)} = \limsup_{n \to \infty} v_1^n.$$

(c) In view of proposition 2.2, we may now conclude from (a), (b) and Theorem 1.3 that $\widetilde{\Gamma}_\phi$ possesses an arbitration value v and that

$$v = j_{\widetilde{\Gamma}}\,(\lim_{k\to\infty} v^{n(k)}) = j_{\widetilde{\Gamma}}\,(\limsup_{n\to\infty} v^n).$$

If $\lim_{k\to\infty} v^{n(k)} \in \overline{\mathcal{W}}(\widetilde{\Gamma})$, then $v_1(\widetilde{\Gamma}_\phi) = \overline{p}_1(\widetilde{\Gamma})$, which implies that $O_1(\widetilde{\Gamma}_\phi) = \widetilde{X} \neq \phi$.

If $\lim_{k\to\infty} v^{n(k)} \in \mathcal{W}(\widetilde{\Gamma}) \cup \mathcal{P}(\widetilde{\Gamma})$, then $\lim_{k\to\infty} v_1^{n(k)} = v_1(\widetilde{\Gamma}_\phi)$ and then (3.6) implies that $\hat{\mu} \in O_1(\widetilde{\Gamma}_\phi)$. Hence, $O_1(\widetilde{\Gamma}_\phi) \neq \phi$. Similarly, one can prove that $O_2(\widetilde{\Gamma}_\phi) \neq \phi$. □

4. Approximation of semi-infinite arbitration games by finite subgames

In the following $\phi : \mathcal{B} \to \Re^2$ is an u.s.c. regular bargaining solution and $A = [a_{ij}]_{i=1,j=1}^{m\ \ \infty}$ and $B = [b_{ij}]_{i=1,j=1}^{m\ \ \infty}$, are bounded $m\times\infty-$ matrices of real numbers. The mixed extension $<S^m, S^\infty, E_1, E_2>$ corresponding to the bimatrix game (A, B), will be denoted by Γ. Here

$$S^m := \left\{ p \in \Re^m;\ p \geqslant 0,\ \sum_{i=1}^m p_i = 1 \right\},$$

$$S^\infty := \left\{ q \in \Re^\infty;\ q \geqslant 0,\ \sum_{j=1}^\infty q_j = 1 \right\},$$

$$E_1(p, q) := pAq = \sum_{i=1}^m \sum_{j=1}^\infty p_i\, a_{ij}\, q_j$$

and

$$E_2(p, q) := pBq, \qquad \text{for all } p \in S^m \text{ and } q \in S^\infty.$$

For each $n \in \mathcal{N}$, we denote by Γ^n the mixed extension $<S^m, S^n, E_1^n, E_2^n>$ of the $m\times n$-bimatrix game $(A^n, B^n) = ([a_{ij}]_{i=1,j=1}^{m\ \ n}, [b_{ij}]_{i=1,j=1}^{m\ \ n})$. By Raiffa's theorem the finite arbitration game Γ_ϕ^n possesses a value, which we denote by $v^n = (v_1^n, v_2^n)$. The question, whether the *semi-infinite arbitration game* Γ_ϕ has a value is answered affirmatively in the next theorem and also the relation with the sequence of values $v^1, v^2, \ldots$ is described. For the proof we need the following lemmas.

Lemma 4.1 : $\lim_{n\to\infty} d_H(\mathcal{R}(\Gamma^n), \mathcal{R}(\Gamma)) = 0.$

Proof: Since $\mathcal{R}_0(\Gamma^1) \subset \mathcal{R}_0(\Gamma^2) \subset \mathcal{R}_0(\Gamma^3), \ldots, \lim_{n\to\infty} \mathcal{R}_0(\Gamma^n)$ exists and equals cl $(\bigcup_{n\in\mathcal{N}} \mathcal{R}_0(\Gamma^n))$ in view of proposition 1 in Salinetti and Wets, (1979, p. 19). By Theorem 2.4.1. in Blackwell and Girshick, (1954, p. 48), for each $q \in S^\infty$ there exists a $\hat{q} \in S^\infty$ with finite carrier such that $\begin{pmatrix} A \\ B \end{pmatrix} q = \begin{pmatrix} A \\ B \end{pmatrix} \hat{q}$, which implies that $\mathcal{R}_0(\Gamma) = \bigcup_{n\in\mathcal{N}} \mathcal{R}_0(\Gamma^n)$. Hence, $\lim_{n\to\infty} d_H(\mathcal{R}_0(\Gamma^n), cl\, \mathcal{R}_0(\Gamma)) = 0$. But then the lemma follows from the inequalities

$$\begin{aligned} 0 &\leqslant d_H(\mathcal{R}(\Gamma^n), \mathcal{R}(\Gamma)) \\ &= d_H(\text{cl conv } \mathcal{R}_0(\Gamma^n), \text{cl conv } \mathcal{R}_0(\Gamma)) \\ &= d_H(\text{conv } \mathcal{R}_0(\Gamma^n), \text{conv cl } \mathcal{R}_0(\Gamma)) \leqslant d_H(\mathcal{R}_0(\Gamma^n), \text{cl } \mathcal{R}_0(\Gamma)). \quad \square \end{aligned}$$

Lemma 4.2 : *Let $\phi : \mathcal{B} \to \mathfrak{R}^2$ be an u.s.c. bargaining solution. Let R be a compact and convex subset of $\mathfrak{R}^2$ and $z, z^1, z^2, z^3\}, \ldots$, a sequence of points in R with $\lim_{n\to\infty} z^n = z$. Then $\lim_{n\to\infty} \phi(z^n, R) = \phi(z\ R)$.*

Proof: This lemma is identical to Corollary 3.1 Jansen and Tijs (1980). $\square$

Theorem 4.3 : *Notation as above.*

(i) *The arbitration game Γ_ϕ possesses a value and $v(\Gamma_\phi) = j_\Gamma\ (\limsup_{n\to\infty} v^n)$.*

(ii) $O_I(\Gamma_\phi) \neq \phi$.

Proof: By Raiffa's theorem, for each $n \in \mathcal{N}$, there exist $\hat{p}^n \in S^m$, $\hat{q}^n \in S^n$ such that

$$\phi_1(E_1^n(\hat{p}^n, q), E_2^n(\hat{p}^n, q)) \geqslant v_1^n, \text{ for all } q \in S^n \qquad \ldots (4.1)$$

$$\phi_2(E_1^n(p, \hat{q}^n), E_2^n(p, \hat{q}^n)) \geqslant v_2^n, \text{ for all } p \in S^m. \qquad \ldots (4.2)$$

Let $t := \limsup_{n\to\infty} v^n$. Take a subsequence $n(1), n(2), \ldots$ of $1, 2, \ldots$ such that

$$\lim_{k\to\infty} v^{n(k)} = t \qquad \ldots (4.3)$$

and such that the sequence $< \hat{p}^{n(k)}; k \in \mathcal{N} >$ converges, say, to $\hat{p}$.

(a) First we prove that $v_1(\Gamma_\phi) \geqslant t_1$. For each $n \in \mathcal{N}$, let $\beta^n : S^\infty \to S^n$ be the map, defined by

$$\beta^n(s_1, s_2, \ldots) = \left(s_1, s_2, \ldots, s_{n-1}, \sum_{j=n}^{\infty} s_j\right), \text{ for each } s \in S^\infty.$$

Take $q \in S^\infty$. Then

$$\lim_{k \to \infty} (\hat{p}^{n(k)} A^{n(k)} \beta^{n(k)}(q), \hat{p}^{n(k)} B^{n(k)} \beta^{n(k)}(q)) = (\hat{p}Aq, \hat{p}Bq). \quad \ldots \quad (4.4)$$

Since ϕ_1 is upper semicontinuous, Lemma 4.1, (4.4), (4.1) and (4.3) imply

$$\phi_1(\hat{p}Aq, \hat{p}Bq) \geqslant \limsup_{k \to \infty} \phi_1(\hat{p}^{n(k)} A^{n(k)} \beta^{n(k)}(q), \hat{p}^{n(k)} B^{n(k)} \beta^{n(k)}(q))$$

$$\geqslant \limsup_{k \to \infty} v_1^{n(k)} = t_1, \text{ for all } q \in S^\infty. \quad \ldots \quad (4.5)$$

This implies that $v_1(\Gamma_\phi) \geqslant t_1$.

(b) Now we prove that $v_2(\Gamma_\phi) \geqslant t_2$. Let $\varepsilon = \frac{1}{3}(t_2 - v_2(\Gamma_\phi))$. We have to show that $\varepsilon \leqslant 0$. Suppose $\varepsilon > 0$. Then, by definition of $v_2(\Gamma_\phi)$, for each $n \in \mathcal{N}$, there exists an $r^n \in S^m$ such that

$$\phi_2(r^n A \alpha^n(\hat{q}^n), r^n B \alpha^n(\hat{q}^n)) \leqslant v_2(\Gamma_\phi) + \epsilon, \quad \ldots \quad (4.6)$$

where $\alpha^n(\hat{q}^n) = (\hat{q}_1^n, \hat{q}_2^n, \ldots, \hat{q}_n^n, 0, 0, \ldots) \in S^\infty$. Denote $(r^n A \alpha^n(\hat{q}^n), r^n B \alpha^n(\hat{q}^n))$ by z^n. Since A and B are bounded matrices, there exists a subsequence $m(1)$, $m(2)$, ... of $n(1)$, $n(2)$, ... such that the sequence $< z^{m(h)}; h \in \mathcal{N} >$ converges, say, to $(u, v) \in \mathcal{R}(\Gamma)$. By Lemma 4.2, (4.6) implies

$$\phi_2(u, v) = \lim_{h \to \infty} \phi_2(z^{m(h)}) \leqslant v_2(\Gamma_\phi) + \varepsilon. \quad \ldots \quad (4.7)$$

On the other hand, for k sufficiently large, we have, in view of (4.2) and (4.3)

$$\phi_2(z^{n(k)}) = \phi_2(r^{n(k)} A^{n(k)} \hat{q}^{n(k)}, r^{n(k)} B^{n(k)} \hat{q}^{n(k)}) \geqslant v_2{}^{n(k)}) \geqslant t_2 - \varepsilon. \quad \ldots \quad (4.8)$$

(Note that in formula (4.7), $\phi_2 = \phi_2^\Gamma$ and in formula (4.8), $\phi_2 = \phi_2^{\Gamma^{n(k)}}$.) Now (4.8), Lemma 4.1 and the u.s.c. of ϕ imply

$$\phi_2(u, v) \geqslant t_2 - \epsilon = v_2(\Gamma_\phi) + 2\varepsilon. \quad \ldots \quad (4.9)$$

But (4.7) and (4.9) are contradictory if $\varepsilon > 0$. Hence, $\varepsilon \leqslant 0$.

(b) In view of Lemma 4.1, Raiffa's theorem and (a) and (b) of this proof, we may conclude, using proposition 2.2, that Γ_ϕ possesses a value and that $v(\Gamma_\phi) = j_\Gamma(\limsup_{n \to \infty} v^n)$.

(c) Now we prove that $O_1(\Gamma_\phi) \neq \phi$. If $t \in \bar{\omega}(\Gamma)$, then $O_1(\Gamma_\phi) = S^m$. If $t \in \mathcal{P}(\Gamma) \bigcup \underline{\omega}(\Gamma)$, then $v_1(\Gamma_\phi) = t_1$ and then $\hat{p} \in O_1(\Gamma_\phi)$ in view of (4.5). $\square$

We conclude with an example of a semi-infinite arbitration game, where

$$\lim_{n \to \infty} v^n \neq v(\Gamma_\phi).$$

Example 4.4 : Let $A = \begin{bmatrix} 1 & 1 & 1 & \cdots \\ \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \cdots \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & \cdots \end{bmatrix}$

Then for each bargaining solution ϕ we have $v(\Gamma_\phi^n) = (1, 0)$ for each $n \in \mathcal{N}$, while $v(\Gamma_\phi) = (1, 1)$.

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